Coding Delay Analysis of Chunked Codes over Line Networks[†]

Anoosheh Heidarzadeh and Amir H. Banihashemi

Department of Systems and Computer Engineering, Carleton University, Ottawa, ON, Canada

Abstract

In this paper, we analyze the coding delay and the average coding delay of Chunked network Codes (CC) over line networks with Bernoulli losses and deterministic regular or Poisson transmissions. Chunked codes are an attractive alternative to random linear network codes due to their lower complexity. Our results, which include upper bounds on the delay and the average delay, are the first of their kind for CC over networks with such probabilistic traffics. These results demonstrate that a stand-alone CC or a precoded CC provides a better tradeoff between the computational complexity and the convergence speed to the network capacity over the probabilistic traffics compared to arbitrary deterministic traffics. The performance of CC over the latter traffics has already been studied in the literature.

I. INTRODUCTION

Chunked codes (CC), originally proposed in [1], generalize random linear network codes (dense codes), and operate by dividing the message of the source into non-overlapping or overlapping sub-messages of equal size, called *chunks* [1]–[3]. Each node at each transmission time randomly chooses a chunk, and transmits it by using a dense code. In fact, a dense code is a CC with only one chunk of the size equal to the message size. Thus, CC require less complex coding operations due to applying coding on chunks smaller than the original message. This however comes at the cost of lower speed of convergence to the capacity compared to dense codes.

The speed of convergence of CC to the capacity of line networks with arbitrary deterministic traffics was studied in [1], [3]. In particular, it has been shown that (i) a CC achieves the capacity, so long as the size of the chunks is lower bounded by a function super-logarithmic in the message size and super-log-cubic in the network length, and (ii) a CC, preceded by a capacity-achieving erasure code, approaches the capacity with an arbitrarily small but non-zero constant gap, so long as the size of the chunks is lower bounded by a function constant in the message size and log-cubic in the network length. There is however no result on the speed of convergence of CC to the capacity over the networks with probabilistic traffics.

The speed of convergence of dense codes to the capacity of some probabilistic traffics was studied in [4], [5]. Very recently, in [6], we studied the coding delay and the average coding delay of a dense code over the traffics

---

[†]This paper is an extended version of an accompanying work submitted to NetCod 2012.

with deterministic regular or Poisson transmissions and Bernoulli losses.[1] The results were in some cases more general, and in some other cases tighter, than the existing bounds in [4], [5].

In this paper, we generalize our analysis in [6], and for the first time, study the coding delay and the average coding delay of CC for different ranges of the chunk sizes.[2]

The main contributions of this work are:

- We derive upper bounds on the coding delay and the average coding delay of a CC alone, or a CC with precoding, over the traffics with deterministic regular transmissions or Poisson transmissions and Bernoulli losses with arbitrary parameters or unequal parameters.

- We show that: (i) a CC achieves the capacity, so long as the size of the chunks is bounded from below by a function super-logarithmic in the message size and super-log-linear in the network length, and (ii) the combination of a CC and a capacity-achieving erasure code approaches the capacity with an arbitrarily small non-zero constant gap, so long as the size of the chunks is bounded from below by a function constant in the message size and log-linear in the network length. The lower bounds in both cases are smaller than those over the networks with arbitrary deterministic traffics. Thus both coding schemes are less computationally complex (require smaller chunks), for the same speed of convergence, over such probabilistic traffics, compared to arbitrary deterministic traffics.

- In a capacity-achieving scenario, for such probabilistic traffics, we show that: (i) the upper bound on the overhead[3] grows sub-log-linearly with the message size and the network length, and decays sub-linearly with the size of the chunks, and (ii) the upper bound on the average overhead grows sub-log-linearly (or poly-log-linearly) with the message size, and sub-log-linearly (or log-linearly) with the network length, and decays sub-linearly (or linearly) with the size of the chunks, in the case with arbitrary (or unequal) parameters. For arbitrary deterministic traffics, the upper bound on the overhead was shown in [3] to be similar to (i), but with a larger (super-linear) growth rate with the network length.

II. NETWORK MODEL AND PROBLEM SETUP

A. Transmission and Loss Model

We consider a unicast problem (one-source one-sink) over a line network with L links connecting $L + 1$ nodes $\{v_i\}_{0 \le i \le L}$ in tandem. The source node v_0 has a message of k vectors (called *message vectors*) from a vector space $\mathcal{F}$ over $\mathbb{F}_2$, and the sink node v_L requires all the message vectors.[4]

---

[1]The *coding delay* of a code over a network with a given traffic (schedule of transmissions and losses) is the minimum time that the code takes to transmit all the message vectors from the source to the sink. The coding delay is a random variable due to the randomness in both the code and the traffic. The *average coding delay* of a code with respect to a class of traffics is the coding delay of the code averaged out over all the traffics (but not the codes), and hence is a random variable due to the randomness in the code.

[2]In this paper, we focus on CC with non-overlapping chunks. The analysis of CC with overlapping chunks is the focus of an ongoing research project.

[3]The (*average*) *overhead* is the difference between the (average) coding delay and the ratio of the message size to the capacity.

[4]The analysis in this paper is generalizable to finite fields of larger size.

Each (non-sink) node at each transmission time transmits a (coded) packet, which is a vector in $\mathcal{F}$. The packet transmissions are assumed to occur in discrete-time, and the transmission times over different links are assumed to follow independent stochastic processes. The transmission times over the ith link are specified by (i) a deterministic process where there is a packet transmission at each time instant, or (ii) a Poisson process with parameter $\lambda_i : 0 < \lambda_i \leq 1$, where λ_i is the average number of transmissions per time unit over the ith link. The transmission schedules resulting from (i) and (ii) are referred to as *deterministic regular* and *Poisson*, respectively.

Each transmitted packet either succeeds or fails to be received (*successful* vs. *lost*). The successful packets are assumed to arrive with zero delay, and the lost packets will never arrive. The packets are assumed to be successful independently over different links. The successful packets over the ith link are specified by a Bernoulli process with (success) parameter $p_i : 0 < p_i \leq 1$, where p_i is the average number of successes per transmission over the ith link. The loss model defined as above is referred to as *Bernoulli*. The special case of Bernoulli loss with all p_i's equal to 1 is analogous to the lossless case.

B. Problem Setup

The goal in this paper is to derive upper bounds on the coding delay and the average coding delay of chunked codes over line networks with deterministic regular or Poisson transmissions and Bernoulli losses.

In a chunked coding scheme, the set of k message vectors at the source node is divided into q disjoint subsets, called *chunks*, each of size $\alpha = k/q$. The source node, at each transmission time, chooses a chunk independently at random, and transmits a packet by randomly linearly combining the message vectors belonging to the underlying chunk. Each non-source non-sink node, at the time of each transmission, chooses a chunk independently at random, and transmits a packet by randomly linearly combining its previously received packets pertaining to the underlying chunk. The global encoding vector[5] of each packet is assumed to be transmitted along with the packet. The sink node can decode a chunk, so long as it receives an innovative[6] collection of packets pertaining to the underlying chunk of a size equal to the size of the chunk.

III. DETERMINISTIC REGULAR TRANSMISSIONS AND BERNOULLI LOSSES

We first review the analysis of dense codes, which are a special case of CC with one chunk, in two cases of arbitrary or unequal (success) parameters, presented in [6].[7] Next, we generalize the analysis to CC with more than one chunk.

---

[5]The *global encoding vector* of a packet is the vector of the coefficients representing the mapping between the message vectors and the packet.

[6]A collection of packets is *innovative* if the global encoding vectors of the packets belonging to the collection are linearly independent.

[7]The details of the proofs in the case of arbitrary parameters were given in [6] and hence omitted. However, neither the details, nor the sketches of the proofs in the case of unequal parameters were given in [6]. We present the sketches of the proofs in this paper for the purpose of completeness.

A. Dense Codes

The goal of the analysis is to lower bound (i) the size of a maximal dense collection of packets at the sink node until a certain time,[8] and then, (ii) the probability that a sufficient number of packets in the underlying collection are innovative.

Let Q_{i+1} and Q_i be the decoding matrices[9] at the $(i+1)^{\text{th}}$ and i^{th} nodes, respectively, and T_i be a matrix over $\mathbb{F}_2$ such that $Q_{i+1} = T_i Q_i$. The entries of Q_{i+1} and Q_i are in $\mathbb{F}_2$. Each row of T_i is the local encoding vectors of a successful packet sent by the ith node. Let Q_i' be Q_i restricted to its rows corresponding to the global encoding vectors of the dense packets at the ith node. Let T_i', the *transfer matrix* at the ith node, be a matrix over $\mathbb{F}_2$ such that $Q_{i+1} = T_i' Q_i'$. Each row of T_i' indicates the labels of the dense packets at the ith node which contribute to a successful packet sent by the ith node.

For every matrix Q over $\mathbb{F}_2$, the *density* of Q, denoted by $\mathcal{D}(Q)$, is the size of a maximal dense collection of rows in Q, where a collection of rows is *dense* if the rows have all independent and uniformly distributed Bernoulli entries. Further, Q is called a *dense matrix* if all its rows form a dense collection. For every matrix T over $\mathbb{F}_2$, the *rank* of T, denoted by $\text{rank}(T)$, is the size of a maximal collection of linearly independent rows in T.

Lemma 1: Let Q be a dense matrix over $\mathbb{F}_2$, and T be a matrix over $\mathbb{F}_2$, where the number of rows in Q and the number of columns in T are equal. If $\text{rank}(T) \geq \gamma$, then $\mathcal{D}(TQ) \geq \gamma$.

Since $Q_{i+1} = T_i' Q_i'$, and Q_i' is dense, $\mathcal{D}(Q_{i+1})$ is lower bounded so long as $\text{rank}(T_i')$ is lower bounded. As shown in [6], the matrix T_i' includes a sub-matrix with the structure of a random block lower-triangular matrix, and the rank of a matrix with such a structure is lower bounded as follows.

Let w, r and $\{r_j\}_{1 \leq j \leq w}$ be arbitrary non-negative integers, and let $r_{\max} = \max_j r_j$ and $r_{\min} = \min_j r_j$. Let $T_{i,j}$ be an $r \times r_j$ dense matrix over $\mathbb{F}_2$, if $1 \leq j \leq i \leq w$; or an arbitrary $r \times r_j$ matrix over $\mathbb{F}_2$, otherwise. Let $T = [T_{i,j}]_{1 \leq i,j \leq w}$. The matrix T is called *random block lower-triangular* (RBLT).

Lemma 2: Let T be an RBLT matrix with parameters w, r and $\{r_j : 0 \leq r_j \leq r\}_{1 \leq j \leq w}$. Let $u = \lceil (n-\gamma)/r_{\min} \rceil$, and $n = \sum_{1 \leq j \leq w} r_j$. For every integer $0 \leq \gamma \leq n-1$, $\Pr\{r(T) < n-\gamma\} \leq u\left(1 - 2^{-r_{\max}}\right) 2^{-\gamma+n-wr+(r-r_{\min})(u-1)}$.

Lemma 3: Let T be an RBLT matrix with parameters w, r and $\{r_j : 0 \leq r \leq r_j\}_{1 \leq j \leq w}$. Let $u = \lceil (n-\gamma)/r \rceil$, and $n = wr$. For every integer $0 \leq \gamma \leq n-1$, $\Pr\{r(T) < n-\gamma\} \leq u\left(1 - 2^{-r}\right) 2^{-\gamma+n-wr_{\min}+(r_{\min}-r)(u-1)}$.

The application of the lemmas is subject to useful and tight choices of w, r, and r_j's. Such parameters depend on the traffic over the ith and $(i+1)^{\text{th}}$ links, and hence not straightforward to optimize. However, by using a probabilistic technique, tight bounds on such parameters can be derived.

Let $(0, N_T]$ be the period of time over which the transmissions occur. Let $(0, N_T]$ be divided into w disjoint partitions of length N_T/w. For every $1 \leq j \leq w$, and $1 \leq i \leq L$, let I_{ij} be the jth partition pertaining to the ith link. For every $i, j: i \leq j \leq w-L+i$, I_{ij} is called *active*. Let $w_T \doteq L(w-L+1)$ be the total number of active

---

[8]A collection of packets is *dense* if the local encoding vectors of the packets are linearly independent, where the local encoding vector of a packet is the vector of the coefficients of the linear combination pertaining to the packet.

[9]The global encoding vectors of the packets at a node form the rows of the *decoding matrix* at the node.

partitions.

Let φ_{ij} be the number of successful packets in I_{ij}. By the assumption, φ_{ij} is a binomial random variable with the expected value $\varphi_i = p_i N_T/w$. Let $p \doteq \min_{1 \leq i \leq L} p_i$, and $\varphi \doteq p N_T/w$. For any real number x, let $\dot{x}$ denote $x/2$. By applying the Chernoff bound, one can show that φ_{ij} is not larger than or equal to $r \doteq (1 - \gamma^*)\varphi$ with probability (w.p.) bounded above by (b.a.b.) $\dot{\epsilon}/w_T$, so long as $0 < \gamma^* < 1$, where $\gamma^* \sim \sqrt{(1/\dot{\varphi})\ln(w_T/\dot{\epsilon})}$. For all i, j, suppose that φ_{ij} is larger than or equal to r.

Let $\mathcal{D}(Q_i^j)$ be the number of dense packets in the first j active partitions over the ith link.

The packets over the first link are all dense. Thus, for all j, $\mathcal{D}(Q_1^j) \geq rj$. For any other values of i, j, by applying Lemma 2, it can be shown that the inequality $\mathcal{D}(Q_i^j) \geq rj - j(1 + o(1))\log(w_T/\epsilon)$ fails w.p. b.a.b. $ij\dot{\epsilon}/w_T$, so long as

$$w \log \frac{w_T}{\epsilon} = o(p N_T). \tag{1}$$

This result shows that the number of dense packets at the sink node, $\mathcal{D}(Q_L)$, fails to be larger than

$$pN_T - O(pN_T L/w) -$$
$$O(\sqrt{pN_T w \log(wL/\epsilon)}) - O(w \log(wL/\epsilon)), \tag{2}$$

w.p. b.a.b. ϵ. By condition (1), it follows that each $O(.)$ term in (2) is $o(pN_T)$ which ensures that the code achieves the capacity. We specify w by $\sqrt[3]{pN_T L^2/\log(pN_T L/\epsilon)}$ in order to maximize (2) subject to condition (1).

Let n_T be equal to (2). Thus, Q_L fails to include an $n_T \times k$ dense sub-matrix w.p. b.a.b. ϵ.

Lemma 4: Let Q be an $n \times k$ ($k \leq n$) dense matrix over $\mathbb{F}_2$. Then, $\Pr\{\text{rank}(Q) < k\} \leq 2^{-(n-k)}$.

By Lemma 4, $\Pr\{\text{rank}(Q_L) < k\}$ is b.a.b. ϵ, so long as $k \leq n_T - \log(1/\epsilon)$. By replacing ϵ with $\dot{\epsilon}$, it follows that the sink node can recover all the message vectors w.p. b.a.b. ϵ, so long as $k \leq n_T - \log(1/\epsilon) - 1$. Let $k_{\max}$ be the largest integer k satisfying this inequality. Thus, $k_{\max} \sim pN_T$, and by replacing N_T with k/p, the following result is immediate.

Theorem 1: The coding delay of a dense code over a line network of L links with deterministic regular traffics and Bernoulli losses with parameters $\{p_i\}$ is larger than

$$\frac{1}{p}\left(k + (1 + o(1))\left(\frac{kL}{w} + \sqrt{k\left(w \log \frac{wL}{\epsilon}\right)} + w \log \frac{wL}{\epsilon}\right)\right)$$

w.p. b.a.b. ϵ, where $w \sim \left(kL^2/\log(kL/\epsilon)\right)^{\frac{1}{3}}$, $p \doteq \min_{1 \leq i \leq L} p_i$.

In the case of the average coding delay, the analysis proceeds by replacing r with φ in the preceding results, and re-specifying w by $\sqrt{pN_T L/\log(pN_T L/\epsilon)}$ in order to maximize

$$pN_T - O(pN_T L/w) - O(w \log(wL/\epsilon)), \tag{3}$$

instead of (2), subject to condition (1).

Theorem 2: The average coding delay of a dense code over a network similar to Theorem 1 is larger than

$$\frac{1}{p}\left(k + (1 + o(1))\left(\frac{kL}{w} + w \log \frac{wL}{\epsilon}\right)\right)$$

w.p. b.a.b. ϵ, where $w \sim (kL/\log(kL/\epsilon))^{\frac{1}{2}}$.

In order to derive tighter bounds the actual values of the success parameters $\{p_i\}$ need to be taken into consideration. In particular, the coding delay and the average coding delay of dense codes for a special case, where no two links have equal success parameters, are upper bounded as follows.

Let us assume $p_1 > p_2 > \cdots > p_L$, without loss of generality. Let $p \doteq \min_{1 \leq i \leq L} p_i$, $\gamma_e \doteq \min_{1 < i \leq L} \gamma_{e_i}$, and $\gamma_{e_i} \doteq |p_i - p_{i-1}|$. Let $r_i \doteq (1 - \gamma_i^*)\varphi_i$, where $\varphi_i = p_i N_T/w$ and $\gamma_i^* \sim \sqrt{(1/\dot\varphi_i)\log(w_T/\dot\epsilon)}$. Let φ_{ij} be defined as before. For all i, j, suppose that φ_{ij} is larger than or equal to r_i.

Similarly as before, for all j, $\mathcal{D}(Q_1^j) \geq r_1 j$. For any other values of i, j, by applying Lemma 3, it can be shown that the inequality $\mathcal{D}(Q_i^j) \geq r_i j$ fails w.p. b.a.b. $ij\dot\epsilon/w_T$, so long as

$$w \log \frac{w_T}{\epsilon} = o(\gamma_e p N_T). \tag{4}$$

Let p, φ, γ^* and r denote p_L, φ_L, γ_L^* and r_L, respectively. Thus, the inequality $\mathcal{D}(Q_L) \geq (1 - \gamma^*)\varphi w_T/L$ fails w.p. b.a.b. ϵ. By replacing φ with $p N_T/w$, the right-hand side of the last inequality can be written as:

$$p N_T - O(p N_T L/w) - O(\sqrt{p N_T w \log(wL/\epsilon)}). \tag{5}$$

The rest of the analysis is similar to that of Theorem 1, except that (5) excludes the last term in (2), and the choice of w needs to satisfy condition (4), instead of condition (1).

Theorem 3: Consider a sequence of unequal parameters $\{p_i\}_{1 \leq i \leq L}$. The coding delay of a dense code over a line network of L links with deterministic regular traffics and Bernoulli losses with parameters $\{p_i\}$ is larger than

$$\frac{1}{p}\left(k + (1 + o(1))\left(\frac{kL}{w} + \sqrt{k\left(w \log \frac{wL}{\epsilon}\right)}\right)\right)$$

w.p. b.a.b. ϵ, where $w \sim \gamma_e \left(kL^2/\log(kL/\epsilon)\right)^{\frac{1}{3}}$, $p \doteq \min_{1 \leq i \leq L} p_i$, $\gamma_e \doteq \min_{1 < i \leq L} \gamma_{e_i}$, and $\gamma_{e_i} \doteq |p_i - p_{i-1}|$.

In the case of the average coding delay, the analysis follows the same line as that of Theorem 2, except that the choice of w needs to maximize

$$p N_T - O(p N_T L/w) \tag{6}$$

subject to condition (4), instead of (3) subject to condition (1).

Theorem 4: The average coding delay of a dense code over a network similar to Theorem 3 is larger than

$$\frac{1}{p}\left(k + (1 + o(1))\left(\frac{kL}{w}\right)\right)$$

w.p. b.a.b. ϵ, where $w \sim \gamma_e k/(f(k) \log(kL/\epsilon))$, and $f(k)$ goes to infinity, as k goes to infinity, such that $f(k) = o(\gamma_e k/\log(kL/\epsilon))$.

B. CC: Capacity-Achieving

In a CC, at each transmission time, a chunk is chosen w.p. $1/q$, and a packet transmission over the ith link is successful w.p. p_i. Thus the probability that a given packet transmission over the ith link is successful and pertains to a given chunk is p_i/q. Thus by replacing p_i with p_i/q in the analysis of dense codes in Section III-A, the coding delay and the average coding delay of CC in a capacity-achieving scenario will be upper bounded as follows.

The results of dense codes are indeed a special case of those of CC with one chunk of size k. It is, however, worth noting that, due to the change in the parameters, the number of partitions w needs to satisfy a new condition: $wq \log \frac{w_T q}{\epsilon} = o(pN_T)$ or $wq \log \frac{w_T q}{\epsilon} = o(\gamma_e pN_T)$, instead of condition (1) or (4), in the proofs of Theorems 5 and 6, or those of Theorems 7 and 8, respectively. Further by replacing w with its optimal choice in the new version of (2), (3), (5) and (6), each $O(.)$ term needs to be $o(pN_T/q)$ in order to ensure that CC are capacity-achieving in the underlying case. Such a condition lower bounds the size of the chunks α by a function super-logarithmic in the message size k.

Theorem 5: The coding delay of a CC with q chunks over a line network of L links with deterministic regular traffics and Bernoulli losses with parameters $\{p_i\}$ is larger than

$$\frac{1}{p}\left(k + (1 + o(1))\left(\frac{kL}{w} + \sqrt{k\left(wq \log \frac{wqL}{\epsilon}\right)} + wq \log \frac{wqL}{\epsilon}\right)\right)$$

w.p. b.a.b. ϵ, so long as $q = o(k/(L \log(kL/\epsilon)))$, where $w \sim \left(kL^2/(q \log(kL/\epsilon))\right)^{\frac{1}{3}}$, and $p \doteq \min_{1 \le i \le L} p_i$.

Proof: The proof follows the same line as in that of Theorem 1 by implementing the following modifications. Let us replace p and ϵ with p/q and ϵ/q, respectively. Then, $\varphi = pN_T/wq$, and $r = (1 - \gamma^*)\varphi$, where $\gamma^* \sim \sqrt{(1/\dot{\varphi}) \ln(w_T q/\dot{\epsilon})}$. Fix a chunk ω. For all $1 \le i \le L$, and $1 \le j \le w - L + 1$, let $\mathcal{D}(Q_i^j)$, $\mathcal{D}_p(Q_i^j)$, and r_{ij} be defined as before, but only restricted to the packets pertaining to the chunk ω. Similarly as before, for all i, j, $\mathcal{D}(Q_i^j)$ can be lower bounded as follows: for all $1 \le j \le w - L + 1$, $\mathcal{D}(Q_1^j) \ge rj$, and for all other values of i, j, $\mathcal{D}(Q_i^j)$ fails to be larger than $rj - j(1 + o(1)) \log(w_T q/\epsilon)$, w.p. b.a.b. $ij\dot{\epsilon}/w_T q$, so long as

$$wq \log \frac{w_T q}{\epsilon} = o(pN_T). \tag{7}$$

Thus the number of dense packets pertaining to the chunk ω at the sink node fails to be larger than

$$\frac{pN_T}{q} - O\left(\frac{pN_T L}{wq}\right) -$$
$$O\left(\sqrt{\frac{pN_T w}{q} \log \frac{wqL}{\epsilon}}\right) - O\left(w \log \frac{wqL}{\epsilon}\right), \tag{8}$$

w.p. b.a.b. ϵ/q. In order to maximize (8) subject to condition (7), we specify w by

$$\sqrt[3]{\frac{pN_T L^2}{q \log(pN_T L/\epsilon)}}.$$

Now let us assume that N_T is $(1 + o(1))k/p$. By replacing ϵ with $\dot{\epsilon}$, in the preceding results, and by replacing k and ϵ with k/q and $\dot{\epsilon}/q$, respectively, in Lemma 4, it follows that the sink node fails to decode the chunk ω w.p. b.a.b. ϵ/q, so long as N_T is larger than

$$\frac{1}{p}\left(k + (1 + o(1))\left(\frac{kL}{w} + \sqrt{k\left(wq \log \frac{wqL}{\epsilon}\right)} + wq \log \frac{wqL}{\epsilon}\right)\right). \tag{9}$$

Taking a union bound over all the chunks, it follows that the sink node fails to decode all the chunks w.p. b.a.b. ϵ, so long as N_T is larger than (9). To ensure that the lower bound on N_T is $(1 + o(1))k/p$, all the terms in (9),

excluding the first one, need to be $o(k/p)$. This condition is met so long as q is

$$o\left(\frac{k}{L\log(kL/\epsilon)}\right).$$ $\qquad\square$

Theorem 6: The average coding delay of a CC with q chunks over a network similar to Theorem 5 is larger than

$$\frac{1}{p}\left(k + (1+o(1))\left(\frac{kL}{w} + wq\log\frac{wqL}{\epsilon}\right)\right)$$

w.p. b.a.b. ϵ, so long as $q = o(k/(L\log(kL/\epsilon)))$, where $w \sim (kL/(q\log(kL/\epsilon)))^{\frac{1}{2}}$.

Proof: The proof is similar to that of Theorem 5, except that r needs to be replaced with φ. This implies that the third term in (8) disappears. Thus by specifying w with

$$\sqrt{\frac{pN_T L}{q\log(pN_T L/\epsilon)}}$$

in order to maximize (8), excluding the third term, subject to condition (7), it follows that the sink node fails to decode all the chunks w.p. b.a.b. ϵ, so long as N_T is larger than

$$\frac{1}{p}\left(k + (1+o(1))\left(\frac{kL}{w} + wq\log\frac{wqL}{\epsilon}\right)\right). \qquad (10)$$

The rest of the proof follows that of Theorem 5. $\qquad\blacksquare$

In the case of unequal success parameters, the coding delay and the average coding delay are upper bounded as follows.

Theorem 7: The coding delay of a CC with q chunks over a line network of L links with deterministic regular traffics and Bernoulli losses with unequal parameters $\{p_i\}$ is larger than

$$\frac{1}{p}\left(k + (1+o(1))\left(\frac{kL}{w} + \sqrt{k\left(wq\log\frac{wqL}{\epsilon}\right)}\right)\right)$$

w.p. b.a.b. ϵ, so long as $q = o\left(\gamma_e^3 k/(L\log(kL/\epsilon))\right)$, where $w \sim \gamma_e\left(kL^2/(q\log(kL/\epsilon))\right)^{\frac{1}{3}}$, $p \doteq \min_{1\le i\le L} p_i$, $\gamma_e \doteq \min_{1 < i \le L} \gamma_{e_i}$, and $\gamma_{e_i} \doteq |p_i - p_{i-1}|$.

Proof: Fix a chunk ω. By replacing p and ϵ with p/q and ϵ/q, respectively, in the proof of Theorem 3, it follows that the number of dense packets pertaining to the chunk ω at the sink node fails to be larger than

$$\frac{pN_T}{q} - O\left(\frac{pN_T L}{wq}\right) -$$
$$O\left(\sqrt{\frac{pN_T w}{q}}\log\frac{wqL}{\epsilon}\right), \qquad (11)$$

w.p. b.a.b. ϵ/q, so long as

$$wq\log\frac{w_T q}{\epsilon} = o(\gamma_e pN_T). \qquad (12)$$

The rest of the proof is similar to that of Theorem 5, except that (11) excludes the last term in (8), and the choice of w needs to satisfy condition (12), instead of condition (7). By specifying w with

$$\sqrt[3]{\frac{\gamma_e^3 pN_T L^2}{q\log(pN_T L/\epsilon)}}$$

in order to maximize (11) subject to condition (12), it follows that the sink node fails to decode all the chunks w.p. b.a.b. ϵ, so long as N_T is larger than

$$\frac{1}{p}\left(k + (1 + o(1))\left(\frac{kL}{w} + \sqrt{k\left(wq\log\frac{wqL}{\epsilon}\right)}\right)\right).\tag{13}$$

In (13), each term, except the largest one, needs to be $o(k/p)$, and this condition is met so long as q is

$$o\left(\frac{\gamma_e^3 k}{L\log(kL/\epsilon)}\right).\qquad\qquad\square$$

Theorem 8: The average coding delay of a CC with q chunks over a network similar to Theorem 7 is larger than

$$\frac{1}{p}\left(k + (1 + o(1))\left(\frac{kL}{w}\right)\right)$$

w.p. b.a.b. ϵ, so long as $q = o\left(\gamma_e k/(f(k)L\log(kL/\epsilon))\right)$, where $w \sim \gamma_e k/(qf(k)\log(kL/\epsilon))$, and $f(k)$ goes to infinity, as k goes to infinity, such that $f(k) = o(\gamma_e k/(\log(kL/\epsilon)))$.

Proof: The proof follows the same line as that of Theorem 5, except that the choice of w needs to maximize

$$\frac{pN_T}{q} - O\left(\frac{pN_T L}{wq}\right)\tag{14}$$

subject to condition (12). To do so, we specify w by

$$\frac{\gamma_e pN_T}{qf(pN_T)\log(pN_T L/\epsilon)},$$

where $f(n)$ goes to infinity, as n goes to infinity, such that $f(n) = o(\gamma_e n/(\log(nL/\epsilon)))$. The sink node fails to decode all the chunks w.p. b.a.b. ϵ, so long as N_T is larger than

$$\frac{1}{p}\left(k + (1 + o(1))\left(\frac{kL}{w}\right)\right).\tag{15}$$

The second term in (15) needs to be $o(k/p)$, and this condition is met so long as q is

$$o\left(\frac{\gamma_e k}{f(k)L\log(kL/\epsilon)}\right).\qquad\qquad\square$$

C. CC with Precoding: Capacity-Approaching with A Gap

By the results of Section III-B, one can conclude that CC are not capacity-achieving if the size of the chunks does not comply with condition $\alpha = \omega(L\log(kL/\epsilon))$.[10] The analysis of Section III-A further does not apply to CC with chunks of small sizes violating the above condition. From a computational complexity perspective, CC with chunks of smaller sizes are, however, of more practical interest (e.g., linear-time CC with constant-size chunks). In the following, we study CC with chunks of a size constant in the message size.

Let $\{p_i\}_{1\leq i\leq L}$ be an arbitrary sequence of success parameters, and let $p \doteq \min_{1\leq i\leq L} p_i$. Let the size of the chunks $\alpha\ (= k/q)$ be a constant in the message size k, i.e., $\alpha = O(1)$. Fix a chunk, and focus on the packets pertaining

---

[10]For non-negative functions $f(n)$ and $g(n)$, we write $f(n) = \omega(g(n))$, if and only if $\lim_{n\to\infty} f(n)/g(n) = \infty$.

to that chunk. Let the time interval $(0, N_T]$ and its w disjoint partitions be defined as before in Section III-A. Let φ_{ij} be the number of packets (pertaining to the given chunk) in the partition I_{ij}, and φ_i be the expected value of φ_{ij}. Let $\varphi \doteq \min_{1 \leq i \leq L} \varphi_i$. Then, $\varphi_i = p_i N_T / wq$, and $\varphi = p N_T / wq$. Let $N_T = (1 + \gamma_c) k / p$, where $0 < \gamma_c < 1$ is an arbitrarily small constant. By replacing N_T with $(1 + \gamma_c) k / p$, $\varphi = (1 + \gamma_c) \alpha / w$, and $\varphi = O(1)$, as w is a constant (otherwise, φ goes to 0, as N_T goes to infinity).

By applying the Chernoff bound, it can be shown that $\Pr\{\varphi_{ij} < (1 - \gamma^*)\varphi\} \leq e^{-\gamma^{*2}\dot{\varphi}}$, for every $0 < \gamma^* < 1$. Taking $e^{-\gamma^{*2}\dot{\varphi}} \leq \dot{\gamma}_b / w_T$, it follows that φ_{ij} is not larger than or equal to $r \doteq (1 - \gamma^*)\varphi$ w.p. b.a.b. $\dot{\gamma}_b / w_T$, where γ^* is the smallest real number satisfying $\gamma^* \geq \sqrt{(1/\dot{\varphi}) \ln(w_T / \dot{\gamma}_b)}$, such that r is an integer ($\gamma^* = O(1)$). Taking a union bound over all the active partitions of all links, it follows that φ_{ij} is not larger than or equal to r w.p. b.a.b. $\dot{\gamma}_b$.

Let $\mathcal{D}(Q_i^j)$ be the number of dense packets pertaining to the given chunk in the first j active partitions over the ith link.

By applying Lemma 3, it can be shown that: (i) for all $1 \leq j \leq w - L + 1$, $\mathcal{D}(Q_1^j) \geq rj$, (ii) for all $1 < i \leq L$, the inequality $\mathcal{D}(Q_i^1) \geq r - \log(w_T / \dot{\gamma}_b)$ fails w.p. b.a.b. $i\dot{\gamma}_b / w_T$, and (iii) for all the other i, j, the inequality $\mathcal{D}(Q_i^j) \geq r - j \log(w_T / \dot{\gamma}_b) - \log((j+1) w_T / \dot{\gamma}_b)$ fails w.p. b.a.b. $ij\dot{\gamma}_b / w_T$, so long as

$$\alpha = \Omega\left(w^2 \log \frac{w_T}{\gamma_b}\right). \tag{16}$$

By using the above results, it follows that the number of dense packets pertaining to the given chunk at the sink node fails to be lower bounded by

$$\frac{w_T \varphi}{L} - O\left(\frac{w_T}{L} \sqrt{\varphi \log \frac{w_T}{\gamma_b}}\right) - O\left(\frac{w_T}{L} \log \frac{w_T}{\gamma_b}\right) \tag{17}$$

w.p. b.a.b. γ_b. The lower bound is non-negative so long as $\alpha = \Omega(w \log(w_T / \gamma_b))$, and this condition holds so long as condition (16) holds. We specify w by $\sqrt[3]{\alpha L^2 / \log(\alpha L / \gamma_b)}$ to maximize (17). By replacing w in (16), it can be rewritten as

$$\alpha = \Omega\left(L^4 \log \frac{L}{\gamma_b}\right). \tag{18}$$

By replacing γ_b with $\dot{\gamma}_b$, and by applying Lemma 4, it follows that the sink node fails to decode the given chunk w.p. b.a.b. γ_b, so long as (17) is larger than $\alpha + \log(1/\dot{\gamma}_b)$. By replacing our choice of w in (17), it can be seen that, excluding the first term, the second term dominates the rest. By replacing φ with $(1 + \gamma_c)\alpha / w$, and by using the properties of the notation $\Omega(.)$, the decoding condition becomes

$$\alpha = \Omega\left(\frac{L}{\gamma_c^3} \log \frac{L}{\gamma_b \gamma_c}\right). \tag{19}$$

Thus, the given chunk is undecodable w.p. b.a.b. γ_b, so long as both conditions (18) and (19) are met. In other words, the expected fraction of undecodable chunks is bounded from above by γ_b. By using a martingale argument similar to the one in [3], the concentration of the fraction of undecodable chunks around the expectation can be shown as follows.

Lemma 5: By applying a CC with chunks of size α, satisfying both conditions (18) and (19), the fraction of undecodable chunks at the sink node until time $N_T = (1 + \gamma_c)k/p$ is larger than $(1 + \gamma_a)\gamma_b$, w.p. b.a.b. ϵ, so long as

$$\alpha^2/\gamma_a^2\gamma_b^2 = o(k/\log(1/\epsilon)), \tag{20}$$

where $0 < \gamma_a, \gamma_b, \gamma_c < 1$ are arbitrary constants.

By the result of Lemma 5, the fraction of chunks which are not decodable until time N_T becomes larger than $(1+\gamma_a)\gamma_b$, w.p. b.a.b. ϵ. Since γ_a, γ_b are non-zero constants, a CC, alone, does not decode all the chunks. However, the completion of decoding of all the chunks is guaranteed by devising a proper precoding scheme [3]. The precoding works as follows: The set of k message vectors at the source node constitute the input of a capacity-achieving (c.-a.) erasure code, called *precode*. The rate of the precode is $1 - (1 + \gamma_a)\gamma_b$ (i.e., the precode decoder can correct up to a fraction $(1+\gamma_a)\gamma_b$ of erasures), and the number of the coded packets at the output of the precode, called *intermediate packets*, is $\left(1 + (1 + \gamma_a)\gamma_b + O(\gamma_b^2)\right)k$. By applying a CC with chunks of size α, satisfying conditions (18), (19) and (20), the fraction of the intermediate packets that are not recoverable at the output of the CC decoder until time $(1 + \gamma_c)\left(1 + (1 + \gamma_a)\gamma_b + O(\gamma_b^2)\right)\frac{k}{p}$ is larger than $(1 + \gamma_a)\gamma_b$, w.p. b.a.b. ϵ. Then, the precode decoder can recover all the k message vectors from the set of recovered intermediate packets. Therefore, the coding delay of a CC with precoding (CCP) is upper bounded as follows.

Theorem 9: The coding delay of a CCP with chunks of size α and a c.-a. erasure code of rate $1 - \gamma_a$, over a line network of L links with deterministic regular traffics and Bernoulli losses with parameters $\{p_i\}$ is larger than $(1 + \gamma_c)\left(1 + (1 + \gamma_a)\gamma_b + O(\gamma_b^2)\right)\frac{k}{p}$, w.p. b.a.b. ϵ, so long as

$$\alpha = \Omega\left(\left\{\left(\frac{L}{\gamma_c^3}\log\frac{L}{\gamma_b\gamma_c}\right), \left(L^4\log\frac{L}{\gamma_b}\right)\right\}\right),$$

and $\alpha^2/\gamma_a^2\gamma_b^2 = o(k/\log(1/\epsilon))$, where $0 < \gamma_a, \gamma_b, \gamma_c < 1$ are arbitrary constants, and $p \doteq \min_{1 \leq i \leq L} p_i$.

In the case of the average coding delay of a CC with precoding, the following can be shown similar to Theorem 9 by replacing r with φ, and hence the proof is omitted.

Theorem 10: The average coding delay of a CCP with chunks of size α and a c.-a. erasure code of rate $1 - \gamma_a$, over a network similar to Theorem 9 is larger than $(1 + \gamma_c)\left(1 + (1 + \gamma_a)\gamma_b + O(\gamma_b^2)\right)\frac{k}{p}$, w.p. b.a.b. ϵ, so long as

$$\alpha = \Omega\left(\frac{L}{\gamma_c}\log\frac{L}{\gamma_b\gamma_c}\right),$$

and $\alpha^2/\gamma_a^2\gamma_b^2 = o(k/\log(1/\epsilon))$, where $0 < \gamma_a, \gamma_b, \gamma_c < 1$ are arbitrary constants.

In the special case of unequal success parameters, the coding delay and the average coding delay of CC with precoding are upper bounded as follows. The proofs follow the same line as in the general case except that a new set of conditions needs to be satisfied based on the assumption that no two success parameters are equal.

Theorem 11: The coding delay of a CCP with chunks of size α and a c.-a. erasure code of rate $1 - \gamma_a$, over a line network of L links with deterministic regular traffics and Bernoulli losses with unequal parameters $\{p_i\}$ is larger than $(1 + \gamma_c)\left(1 + (1 + \gamma_a)\gamma_b + O(\gamma_b^2)\right)\frac{k}{p}$, w.p. b.a.b. ϵ, so long as

$$\alpha = \Omega\left(\left\{\left(\frac{L}{\gamma_c^3}\log\frac{L}{\gamma_b\gamma_c}\right), \left(\frac{L}{\gamma_e^3}\log\frac{L}{\gamma_e\gamma_b}\right)\right\}\right),$$

TABLE I

COMPARISON OF CHUNKED CODES OVER LINE NETWORKS WITH VARIOUS TRAFFICS

Traffic	Success Parameters	Overhead (η) and Average Overhead ($\bar{\eta}$)	Size of Chunks (α)	w	Comments
Arbitrary Deterministic	-	$\eta = \bar{\eta} = O\left(kL\left(\frac{1}{\alpha}\log\frac{kL}{\epsilon}\right)^{\frac{1}{3}}\right)$	$\omega\left(L^3\log\frac{kL}{\epsilon}\right)$	-	$m = \frac{kw}{\alpha}\log\left(\frac{kLw}{\alpha\epsilon}\right)$ $f(k) = o\left(\frac{\gamma_e k}{\log\frac{kL}{\epsilon}}\right)$ $\lim_{k\to\infty}f(k) = \infty$ $\gamma_{e_i} = |p_i - p_{i-1}|$ $\gamma_e = \min_{1<i\leq L}\gamma_{e_i}$ $p = \min_{1\leq i\leq L}p_i$
Deterministic Regular Transmissions and Bernoulli Losses	Arbitrary	$\eta = \frac{1}{p}\left((1+o(1))\left(\frac{kL}{w}+k^{\frac{1}{2}}m^{\frac{1}{2}}+m\right)\right)$ $\bar{\eta} = \frac{1}{p}\left((1+o(1))\left(\frac{kL}{w}+m\right)\right)$	$\omega\left(L\log\frac{kL}{\epsilon}\right)$	$\left(\frac{\alpha L^2}{\log\frac{kL}{\epsilon}}\right)^{\frac{1}{3}}$ $\left(\frac{\alpha L}{\log\frac{kL}{\epsilon}}\right)^{\frac{1}{2}}$	
	Unequal	$\eta = \frac{1}{p}\left((1+o(1))\left(\frac{kL}{w}+k^{\frac{1}{2}}m^{\frac{1}{2}}\right)\right)$ $\bar{\eta} = \frac{1}{p}\left((1+o(1))\left(\frac{kL}{w}\right)\right)$	$\omega\left(\frac{L}{\gamma_e^3}\log\frac{kL}{\epsilon}\right)$ $\omega\left(f(k)\left(\frac{L}{\gamma_e}\log\frac{kL}{\epsilon}\right)\right)$	$\left(\frac{\gamma_e^3\alpha L^2}{\log\frac{kL}{\epsilon}}\right)^{\frac{1}{3}}$ $\frac{1}{f(k)}\left(\frac{\gamma_e\alpha}{\log\frac{kL}{\epsilon}}\right)$	

TABLE II

COMPARISON OF CHUNKED CODES WITH PRECODING (A CAPACITY-ACHIEVING ERASURE CODE) OVER LINE NETWORKS WITH VARIOUS TRAFFICS

Traffic	Success Parameters	Overhead (η) and Average Overhead ($\bar{\eta}$)	Size of Chunks (α)		Comments
Arbitrary Deterministic	-	$\eta = \bar{\eta} = \gamma_o k$	$\Omega\left(\frac{L^3}{\gamma_c^3}\log\frac{L}{\gamma_b\gamma_c}\right)$		$0 < \gamma_a, \gamma_b, \gamma_c < 1$ $\{\gamma_a,\gamma_b,\gamma_c\} = O(1)$ $\gamma_o = \gamma_c + (1+\gamma_c)\gamma_o'$ $\gamma_o' = (1+\gamma_a)\gamma_b + O(\gamma_b^2)$ $\gamma_{e_i} = |p_i - p_{i-1}|$ $\gamma_e = \min_{1<i\leq L}\gamma_{e_i}$ $p = \min_{1\leq i\leq L}p_i$
Deterministic Regular Transmissions and Bernoulli Losses	Arbitrary	$\eta = \gamma_o\frac{k}{p}$	$\Omega\left(\left\{\left(\frac{L}{\gamma_c^3}\log\frac{L}{\gamma_b\gamma_c}\right),\left(L^4\log\frac{L}{\gamma_b}\right)\right\}\right)$	$o\left(\sqrt{\frac{\gamma_a^2\gamma_b^2 k}{\log\frac{1}{\epsilon}}}\right)$	
		$\bar{\eta} = \gamma_o\frac{k}{p}$	$\Omega\left(\frac{L}{\gamma_c}\log\frac{L}{\gamma_b\gamma_c}\right)$		
	Unequal	$\eta = \gamma_o\frac{k}{p}$	$\Omega\left(\left\{\left(\frac{L}{\gamma_c^3}\log\frac{L}{\gamma_b\gamma_c}\right),\left(\frac{L}{\gamma_e^3}\log\frac{L}{\gamma_b\gamma_e}\right)\right\}\right)$		
		$\bar{\eta} = \gamma_o\frac{k}{p}$	$\Omega\left(\frac{L}{\gamma_e^2\gamma_c}\log\frac{L}{\gamma_b\gamma_c}\right)$		

and $\alpha^2/\gamma_a^2\gamma_b^2 = o(k/\log(1/\epsilon))$, where $0 < \gamma_a, \gamma_b, \gamma_c < 1$ are arbitrary constants, $p \doteq \min_{1\leq i\leq L}p_i$, $\gamma_e \doteq \min_{1<i\leq L}\gamma_{e_i}$, and $\gamma_{e_i} \doteq |p_i - p_{i-1}|$.

Proof: Let us assume $p_1 > p_2 > \cdots > p_L$, without loss of generality. Let $p \doteq \min_{1\leq i\leq L}p_i$, $\gamma_e \doteq \min_{1<i\leq L}\gamma_{e_i}$, and $\gamma_{e_i} \doteq |p_i - p_{i-1}|$. Fix a chunk. Let $r_i \doteq (1-\gamma_i^*)\varphi_i$, where $\varphi_i = p_i N_T/wq$ and $\gamma_i^* \sim \sqrt{(1/\dot{\varphi}_i)\log(w_T/\dot{\gamma_b})}$, and $0 < \gamma_b < 1$ is an arbitrary constant. Let φ_{ij} be the number of packets (pertaining to the given chunk) in the partition I_{ij} (the jth partition pertaining to the ith link), where the time interval $(0, N_T]$ is split into w partitions of length N_T/w, and let φ_i be the expected value of φ_{ij}. For all i, j, suppose that φ_{ij} is larger than or equal to r_i. Let $N_T = (1+\gamma_c)k/p$, where $0 < \gamma_c < 1$ is an arbitrarily small constant. By replacing N_T with $(1+\gamma_c)k/p$, $\varphi_i = (1+\gamma_c)p_i\alpha/pw$, and $\varphi = O(1)$, similar to that in the proof of Theorem 9.

Similarly as before, for all j, $\mathcal{D}(Q_1^j) \geq r_1 j$. For any other values of i, j, by applying Lemma 3, it can be shown that the inequality $\mathcal{D}(Q_i^j) \geq r_i j$ fails w.p. b.a.b. $ij\dot{\gamma_b}/w_T$, so long as

$$\alpha = \Omega\left(\frac{w}{\gamma_e^2}\log\frac{w_T}{\gamma_b}\right). \tag{21}$$

Let φ, γ^* and r denote φ_L, γ_L^* and r_L, respectively. Thus, the number od dense packets pertaining to the given chunk at the sink node fails to be larger than

$$\alpha(1 + \gamma_c) - O\left(\frac{\alpha L}{w}\right) - O\left(\sqrt{\alpha w \log \frac{w_T}{\gamma_b}}\right). \tag{22}$$

We specify w by

$$\left(\frac{\alpha L^2}{\log(w_T/\gamma_b)}\right)^{\frac{1}{3}}$$

to maximize (22) subject to condition (21). For this choice of w, condition (21) is met so long as

$$\alpha = \Omega\left(\frac{L}{\gamma_e^3} \log \frac{L}{\gamma_e \gamma_b}\right). \tag{23}$$

By replacing γ_b with $\dot{\gamma}_b$ in the preceding results, and substituting w in (22), the result of Lemma 4 shows that the sink node fails to decode the given chunk w.p. b.a.b. γ_b, so long as (22) is larger than $\alpha + \log(1/\dot{\gamma}_b)$. Based on the properties of the notation $\Omega(.)$, the latter condition is met so long as

$$\alpha = \Omega\left(\frac{L}{\gamma_c^3} \log \frac{L}{\gamma_b \gamma_c}\right). \tag{24}$$

The rest of the proof is similar to the proof of Theorem 9, except that in this case conditions (23) and (24) need to be met, instead of conditions (18) and (19). ∎

Theorem 12: The average coding delay of a CCP with chunks of size α and a c.-a. erasure code of rate $1 - \gamma_a$, over a network similar to Theorem 11 is larger than $(1 + \gamma_c)\left(1 + (1 + \gamma_a)\gamma_b + O(\gamma_b^2)\right)\frac{k}{p}$, w.p. b.a.b. ϵ, so long as

$$\alpha = \Omega\left(\frac{L}{\gamma_e^2 \gamma_c} \log \frac{L}{\gamma_b \gamma_c}\right),$$

and $\alpha^2/\gamma_a^2 \gamma_b^2 = o(k/\log(1/\epsilon))$, where $0 < \gamma_a, \gamma_b, \gamma_c < 1$ are arbitrary constants.

Proof: The proof follows the same line as that of Theorem 11, except that the choice of w needs to maximize

$$\alpha(1 + \gamma_c) - O\left(\frac{\alpha L}{w}\right) \tag{25}$$

subject to condition (21). To do so, the choice of w needs to be $\Omega(L/\gamma_c)$, and hence, condition (21) becomes

$$\alpha = \Omega\left(\frac{L}{\gamma_e^2 \gamma_c} \log \frac{L}{\gamma_b \gamma_c}\right). \qquad \square$$

IV. POISSON TRANSMISSIONS AND BERNOULLI LOSSES

In the case of Bernoulli losses and Poisson transmissions with parameters $\{p_i\}_{1 \leq i \leq L}$ and $\{\lambda_i\}_{1 \leq i \leq L}$, the points in time at which the arrivals/departures occur over the ith link follow a Poisson process with parameter $\lambda_i p_i$. Thus the number of packets pertaining to a given chunk, in each partition pertaining to the ith link, has a Poisson distribution with the expected value $\lambda_i p_i N_T / wq$. Since the result of Chernoff bound also holds for Poisson random variables, the main results in Section III apply to this case by replacing p with λp, where $\{\lambda, p\} \doteq \{\lambda_\mu, p_\mu\}$, and $\mu \doteq \arg\min_{1 \leq i \leq L} \lambda_i p_i$.

V. Discussion

Table I shows the upper bounds[11] (w.p. of failure b.a.b. ϵ) on the overhead and the average overhead of CC over various traffics for different ranges of the size of the chunks based on the results in Section III and those in [3].[12] The traffics are: arbitrary deterministic traffics, or traffics with deterministic regular transmissions and Bernoulli losses. We refer to the latter traffics as the *probabilistic traffics* for simplifying the terminology. The probabilistic traffics are categorized into two sub-categories: traffics with arbitrary success parameters and traffics with unequal success parameters. In the case of arbitrary deterministic traffics, the capacity is 1, and in the case of probabilistic traffics with success parameters $\{p_i\}_{1 \leq i \leq L}$, the capacity is p, where $p = \min_{1 \leq i \leq L} p_i$. We say that a code is "capacity-achieving" (c.-a.) if the ratio of the overhead to k/p goes to 0, as k goes to infinity. Similarly, a code is "capacity-achieving on average" (c.-a.a.) if the ratio of the average overhead to k/p goes to 0, as k goes to infinity. In Table I, the upper (or lower) row in front of each case of success parameters corresponds to a c.-a. (or a c.-a.a.) scenario.

In the table, one can see that, for each traffic, the size of the chunks (α) has to be sufficiently large so that CC are c.-a. or c.-a.a.. For arbitrary deterministic traffics, the lower bound on α is super-logarithmic in k, i.e., $\omega(\log k)$, and super-log-cubic in L, i.e., $\omega(L^3 \log L)$. For the probabilistic traffics with arbitrary or unequal success parameters, the lower bound on α has a similar growth rate with k, but a smaller (super-log-linear) growth rate with L, i.e., $\omega(L \log L)$. The coding cost of CC (i.e., the ratio of the number of the coding (packet) operations to k), is, on the other hand, linear in α. Thus, CC can perform as fast over both the arbitrary deterministic traffics and the probabilistic traffics, but with a lower coding cost (smaller chunks) in the latter case compared to the former.

Moreover, as it can be seen in Table I, for both arbitrary deterministic and probabilistic traffics (in each case of arbitrary or unequal success parameters), the overhead grows sub-log-linearly with k, i.e., $O(k \log^{\frac{1}{3}} k)$, and decays sub-linearly with α, i.e., $O(1/\alpha^{\frac{1}{3}})$. However, for arbitrary deterministic traffics, the overhead grows with $O(L \log^{\frac{1}{3}} L)$, and for the probabilistic traffics, it only grows with $O(L^{\frac{1}{3}} \log^{\frac{1}{3}} L)$. This implies a faster speed of convergence to the capacity in the latter case compared to the former. Similar comparison result can also be observed in terms of the average overhead, except that in the case of unequal success parameters, the average overhead decays linearly with α, i.e., $O(1/\alpha)$, but grows poly-log-linearly with k, i.e., $O(k \log^2 k)$, for the choice of $f(k) = O(\gamma_e \log k)$, and log-linearly with L, i.e., $O(L \log L)$.

Table II also shows the results for CC with precoding (CCP) in the scenarios similar to those considered in Table I, where the precode is an (capacity-achieving) erasure code of dimension k and rate $1 - \gamma_a$. In particular, one can see that CCP are "capacity-approaching" or "capacity-approaching on average" with an arbitrary small "non-zero constant" gap γ_o (i.e., the ratio of the overhead or the average overhead to k/p goes to γ_o, as k goes to infinity) if α is sufficiently large. For simplifying the terminology, we drop the term "with a non-zero constant

---

[11]With a slight abuse of language, we refer to the "upper bound" on the overhead or the average overhead as the "overhead" or the "average overhead."

[12]The results of Section III-B and those of Section III-C were stated in terms of q and α, respectively. In this section, for the ease of comparison, the former results are also restated in terms of α by replacing q with k/α.

gap." The upper row (or the lower row) in front of each case of success parameters corresponds to a capacity-approaching (or a capacity-approaching on average) scenario. For arbitrary deterministic traffics, the lower bound on α is constant in k, and log-cubic in L, i.e., $O(L^3 \log L)$. For the probabilistic traffics with arbitrary or unequal success parameters, the lower bound on α is also constant in k, but has a smaller (log-linear) growth rate with L, i.e., $O(L \log L)$. Thus, in the case of CCP, one can make a conclusion similar to the one made in the case of stand-alone CC, with respect to the arbitrary deterministic and the probabilistic traffics.

REFERENCES

[1] P. Maymounkov, N. Harvey, and D. Lun, "Methods for Efficient Network Coding," in *Proc. 44th Annual Allerton Conference on Communication Control and Computing*, 2006, pp. 482–491.

[2] D. Silva, W. Zeng, and F. Kschischang, "Sparse network coding with overlapping classes," in *Network Coding, Theory, and Applications, 2009. NetCod '09. Workshop on*, June 2009, pp. 74 –79.

[3] A. Heidarzadeh and A. Banihashemi, "Network Codes with Overlapping Chunks over Line Networks: A Case for Linear-Time Codes," *Submitted to IEEE Trans. Info. Theory*, May 2011. [Online]. Available: http://arxiv.org/abs/1105.5736

[4] P. Pakzad, C. Fragouli, and A. Shokrollahi, "Coding Schemes for Line Networks," in *Proc. IEEE Int. Symp. Info. Theory, ISIT'05*, 2005.

[5] T. Dikaliotis, A. Dimakis, T. Ho, and M. Effros, "On the Delay of Network Coding over Line Networks," in *Proc. IEEE Int. Symp. Info. Theory, ISIT'09*, 28 2009-july 3 2009, pp. 1408–1412.

[6] A. Heidarzadeh and A. Banihashemi, "How Fast Can Dense Codes Achieve the Min-Cut Capacity of Line Networks?" *Submitted to IEEE ISIT'12*, Jan. 2012. [Online]. Available: http://arxiv.org/abs/1202.0343